Worksheet 5 Memorandum: Trigonometry (Compound Angles)

Grade 12 Mathematics CAPS

1.

a) $\cos(x+y) = \cos x \cos y - \sin x \sin y$

b) $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$

c) $\cos 2c = 2\cos^2 c - 1$

d) $\cos(b-c) = \cos b \cos c + \sin b \sin c$

e) $\sin 2x = 2\sin x \cos x$

f) $\sin(y-z) = \sin y \cos z - \sin z \cos y$

g) $\cos 2d = 1 - 2\sin^2 d$

h) $\cos 2e = \cos^2 e - \sin^2 e$

2.

a) $\sin(75°)$

$= \sin(30° + 45°)$

$= \sin 30° \cos 45° + \sin 45° \cos 30°$

$= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$

b) $\sin(105°)$

$= \sin(60° + 45°)$

$= \sin 60° \cos 45° + \sin 45° \cos 60°$

$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$

$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$

c) $\sin(120°)$

$= \sin(90° + 30°)$ or $\sin(60° + 60°)$

$= \sin 90° \cos 30° + \cos 90° \sin 30°$

$= 1\left(\frac{\sqrt{3}}{2}\right) + 0\left(\frac{1}{2}\right)$

$= \frac{\sqrt{3}}{2}$

d) $\sin(135°)$

$= \sin(90° + 45°)$

$= \sin 90° \cos 45° + \sin 45° \cos 90°$

$= (1)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)(0)$

$= \frac{\sqrt{2}}{2}$

e) $\cos(150°)$

$= \cos(90° + 60°)$

$= \cos 90° \cos 60° - \sin 60° \sin 90°$

$= 0\left(\frac{1}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)(1)$

$= -\frac{\sqrt{3}}{2}$

f) $\cos(15°)$

$= \cos(45° - 30°)$

$= \cos 45° \cos 30° + \sin 45° \sin 30°$

$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$

$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$

g) $\cos(135°)$

$= \cos(90° + 45°)$

$= \cos 90° \cos 45° - \sin 90° \sin 45°$

$= 0\left(\frac{\sqrt{2}}{2}\right) - 1\left(\frac{\sqrt{2}}{2}\right)$

$= -\frac{\sqrt{2}}{2}$

h) $\cos(75°)$

$= \cos(45° + 30°)$

$= \cos 45° \cos 30° - \sin 45° \sin 30°$

$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$

3. a) $\cos 23°\cos 22° - \sin 23° \sin 22°$

$= \cos(23° + 22°)$

$= \cos 45°$

b) $1 - 2\sin 45° \cos 45°$

$= 1 - 2\sin 45° \sin 45°$

$= 1 - 2\sin^2 45°$

$= \cos 2(45°)$

$= \cos 90°$

c) $\sin 18° \cos 19° + \sin 19° \cos 18°$

$= \sin(18° + 19°)$

$= \sin 37°$

d) $2\cos^2 A - 1$

$= \cos 2A$

e) $\cos 81° \cos 11° + \sin 81° \cos 11°$

$= \cos(81° - 11°)$

$= \cos 70°$

f) $2\sin 16°\cos 16°$

$= \sin 2(16°)$

$= \sin 32°$

g) $\cos 25°\cos 25° - \sin 25° \sin 25°$

$= \cos^2 25° - \sin^2 25°$

$= \cos 2(25°)$

$= \cos 50°$

h) $\sin 28°\cos 8° - \sin 8° \cos 28°$

$= \sin(28° - 8°)$

$= \sin 20°$

4. a) $\cos(a + a) = 1 - 2\sin^2 a$

LHS $= \cos a \cos a - \sin a \sin a$

$= \cos^2 a - \sin^2 a$

$= (1 - \sin^2 a) - \sin^2 a$

$= 1 - 2\sin^2 a$

∴LHS = RHS

b) $\frac{\cos 2b + 1}{\sin 2b} = \cot b$

LHS $= \frac{2\cos^2 b - 1 + 1}{2\sin b \cos b}$

$= \frac{2\cos^2 b}{2\sin b \cos b}$

$= \frac{\cos b}{\sin b}$

$= \cot b$

∴ LHS = RHS

c) $\sin(d+45°) - \cos(d+45°) = \sqrt{2}\sin d$

LHS $= \sin d\cos 45° + \sin 45°\cos d - (\cos d\cos 45° - \sin d\sin 45°)$

$= \frac{\sqrt{2}}{2}\sin d + \frac{\sqrt{2}}{2}\cos d - \frac{\sqrt{2}}{2}\cos d + \frac{\sqrt{2}}{2}\sin d$

$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)\sin d$

$= \sqrt{2}\sin d$

∴LHS = RHS

d) $\frac{1-\sin 2w}{\sin 2w - \cos 2w - 1} = \frac{1}{2}\tan w - \frac{1}{2}$

LHS $= \frac{1-2\sin w\cos w}{2\sin w\cos w - (2\cos^2 w - 1) - 1}$

$= \frac{\cos^2 w + \sin^2 w - 2\sin w\cos w}{2\sin w\cos w - 2\cos^2 w + 1 - 1}$

$= \frac{\sin^2 w - 2\sin w\cos w + \cos^2 w}{2\sin w\cos w - 2\cos^2 w}$

$= \frac{(\sin w - \cos w)(\sin w - \cos w)}{2\cos w(\sin w - \cos w)}$

$= \frac{\sin w - \cos w}{2\cos w}$

$= \frac{\sin w}{2\cos w} - \frac{\cos w}{2\cos w}$

$= \frac{1}{2}\tan w - \frac{1}{2}$ ∴ LHS = RHS

e) $\frac{\sin(b+30°) + \cos(60°+b)}{\sin 2b} = \frac{1}{2}cosec\ b$

LHS $= \frac{\sin b\cos 30° + \cos b\sin 30° + \cos 60°\cos b - \sin 60°\sin b}{2\sin b\cos b}$

$= \frac{\frac{\sqrt{3}}{2}\sin b + \frac{1}{2}\cos b + \frac{1}{2}\cos b - \frac{\sqrt{3}}{2}\sin b}{2\sin b\ \cos b}$

$= \frac{cosb}{2\sin b\cos b}$

$= \frac{1}{2\sin b}$

$= \frac{1}{2}cosec\ b$

∴ LHS = RHS

f) $\sin 3a + \cos 3a = (\cos a - \sin a)(1 + 4\cos a\sin a)$

LHS $= \sin(2a + a) + \cos(2a + a)$

$= \sin 2a\cos a + \cos 2a\sin a + \cos 2a\cos a - \sin 2a\sin a$

$= \cos a\,(2\sin a\cos a) + \sin a\,(\cos^2 a - \sin^2 a) + \cos a\,(\cos^2 a - \sin^2 a) - \sin a\,(2\sin a\cos a)$

$= 2\sin a\cos^2 a + \cos^2 a\sin a - \sin^3 a + \cos^3 a - \sin^2 a\cos a - 2\sin^2 a\cos a$

$= 3\sin a\cos^2 a + \cos^3 a - \sin^3 a - 3\sin^2 a\cos a$

$= (3\sin a\cos^2 a - 3\sin^2 a\cos a) + (\cos^3 a - \sin^3 a)$

$= 3\sin a\cos a\,(\cos a - \sin a) + (\cos a - \sin a)(\cos^2 a + \cos a\,.\sin a + \sin^2 a)$

$= 3\sin a\cos a\,(\cos a - \sin a) + (\cos a - \sin a)(1 + \cos a\sin a)$

$= (\cos a - \sin a)(3\sin a\cos a + 1 + \sin a\ \cos a)$

$= (\cos a - \sin a)(1 + 4\cos a\sin a)$ ∴ LHS = RHS

g) $\frac{\cos 2f-1}{\sin 2f} = -\tan f$

LHS $= \frac{1-2\sin^2 f-1}{2\sin f\cos f}$

$= \frac{-2\sin^2 f}{2\sin f\cos f}$

$= \frac{-\sin f}{\cos f}$

$= -\tan f$

∴ LHS = RHS

h) $\frac{1-2\sin d}{\cos d-\sin d} - \frac{2\cos d-1}{\cos d+\sin d} = \frac{2(\cos d-1)}{\cos 2d}$

RHS $= \frac{2\cos d-2}{\cos^2 d-\sin^2 d}$

LHS $= \frac{(1-2\sin d)(\cos d+\sin d)-(2\cos d-1)(\cos d-\sin d)}{(\cos d-\sin d)(\cos d+\sin d)}$

$= \frac{\cos d+\sin d-2\sin d\cos d-2\sin^2 d-(2\cos^2 d-2\sin d\cos d-\cos d+\sin d)}{\cos^2 d-\sin^2 d}$

$= \frac{\cos d+\sin d-2\sin d\cos d-2\sin^2 d-2\cos^2 d+2\sin d\cos d+\cos d-\sin d}{\cos^2 d-\sin^2 d}$

$= \frac{2\cos d-2\sin^2 d-2\cos^2 d}{\cos^2 d-\sin^2 d}$

$= \frac{2\cos d-2(\sin^2 d+\cos^2 d)}{\cos^2 d-\sin^2 d}$

$= \frac{2\cos d-2}{\cos^2 d-\sin^2 d}$

∴LHS = RHS

i) $\frac{1}{1-\sin A} + \frac{2\sin A}{1+\sin A} = 2 - \frac{1-3\sin A}{\cos^2 A}$

RHS $= \frac{2\cos^2 A-1+3\sin A}{\cos^2 A}$

$= \frac{\cos 2A+3\sin A}{\cos^2 A}$

LHS $= \frac{1+\sin A+(1-\sin A)(2\sin A)}{(1-\sin A)(1+\sin A)}$

$= \frac{1+\sin A+2\sin A-2\sin^2 A}{1-\sin^2 A}$

$= \frac{1-2\sin^2 A+3\sin A}{\cos^2 A}$

$= \frac{\cos 2A+3\sin A}{\cos^2 A}$

∴ LHS = RHS

j) $1 - \frac{\sin 2B+1}{\cos 2B} = \frac{2\sin B}{\sin B-\cos B}$

LHS $= \frac{\cos 2B-\sin 2B-1}{\cos 2B}$

$= \frac{1-2\sin^2 B-2\sin B\cos B-1}{\cos^2 B-\sin^2 B}$

$= \frac{-2\sin^2 B-2\sin B\cos B}{(\cos B-\sin B)(\cos B+\sin B)}$

$= \frac{-2\sin B(\sin B+\cos B)}{(\cos B-\sin B)(\cos B+\sin B)}$

$= \frac{-2\sin B}{\cos B-\sin B}$

$= \frac{2\sin B}{\sin B-\cos B}$

∴ LHS = RHS

5. a) $\sin a\sin b + \cos a\cos b = 0.5$ where $a = 2b$

$\cos(a+b) = 0.5$

$\cos(2b+b) = 0.5$

$\cos 3b = 0.5$

$3b = 60° + k360°$ OR $3b = 360° - 60° + k.360°$

$b = 20° + k120°$ $b = 100° + k.120°$

$\therefore a = 40° + k240°$ $a = 200° + k240°$

b) $\sin 2c - \sin c = 0$

$2\sin c\cos c - \sin c = 0$

$\sin c\,(2\cos c - 1) = 0$

$\therefore \sin c = 0 \quad or \quad 2\cos c = 1$

$\therefore c = 0 + k.360°$

OR $c = 180° + k360°$

OR $\cos c = \frac{1}{2}$

$\therefore c = 60° + k.360°$

c) $\cos 2d + \cos d = 0$

$2\cos^2 d - 1 + \cos d = 0$

$2\cos^2 d + \cos d - 1 = 0$

$(2\cos d - 1)(\cos d + 1) = 0$

$\cos d = \frac{1}{2}$ $\qquad \cos d = -1$

$\therefore d = 60° + k.360°$

OR $d = 300° + k360°$

OR $d = 180° - 0° + k.360°$

$d = 180° + k.360°$

d) $\cos 3d\sin 4d - \sin 3d\cos 4d = 0.33$

$\sin(4d - 3d) = 0.33$

$\sin d = 0.33$

$d = 19.27° + k.360°$

OR $d = 180° - 19.27° + k360°$

$d = 160.73° + k360°$

e) $\sin(\alpha + 30°) = \cos a$

$\sin(a + 30°) = \sin(90° - a)$

$a + 30° = 90° - a$

$2a = 60° + k.360$

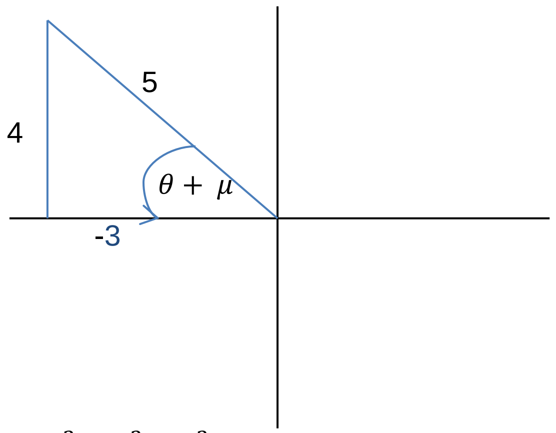
$a = 30° + k180°$

OR $2a = 180° - 60° + k\,360°$

$a = 60° + k180°$

6. $5\sin(\theta + \mu) = 4,\ \tan(\theta + \mu) < 0$

$\sin(\theta + \mu) = \frac{4}{5}$ $\qquad$ tan negative



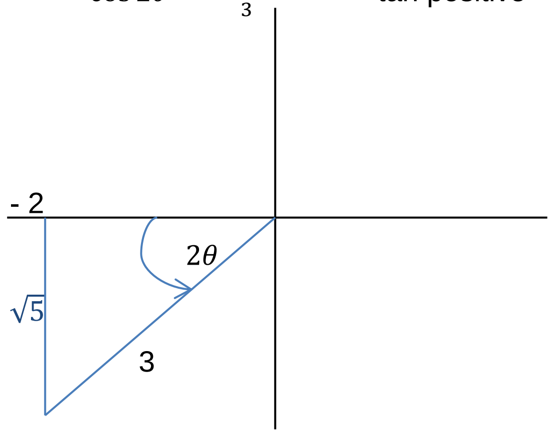
$x^2 = r^2 - y^2$

$x = \sqrt{(5)^2 - (4)^2}$

$x = -3$

$3\cos 2\theta + 2 = 0,\ \tan 2\theta > 0$

$\cos 2\theta = -\frac{2}{3}$ $\qquad$ tan positive



$y^2 = r^2 - x^2$

$y = \sqrt{(3)^2 - (-2)^2}$

$y = \sqrt{5}$

a) $\sin\theta$

$\cos 2\theta = \frac{-2}{3}$

$1 - 2\sin^2\theta = -\frac{2}{3}$

$-2\sin^2\theta = -1\frac{2}{3}$

$\sin^2\theta = \frac{5}{6}$

$\sin\theta = \sqrt{\frac{5}{6}}$

b) $\cos\theta$

$\cos 2\theta = -\frac{2}{3}$

$2\cos^2\theta - 1 = -\frac{2}{3}$

$2\cos^2\theta = \frac{1}{3}$

$\cos^2\theta = \frac{1}{6}$

$\cos\theta = \sqrt{\frac{1}{6}}$

c) $\tan(\theta + \mu)$

$= \frac{4}{3}$

d) $\cos\mu$

$\sin(\theta + \mu) = \frac{4}{5}$ and $\cos(\theta + \mu) = \frac{-3}{5}$

$\sin\theta\cos\mu + \sin\mu\cos\theta = \frac{4}{5}$

$\cos\theta\cos\mu - \sin\theta\sin\mu = \frac{-3}{5}$ … 2

$\sin\mu\cos\theta = \frac{4}{5} - \sin\theta\cos\mu$

$\sin\mu = \frac{\left(\frac{4}{5} - \left(\sqrt{\frac{5}{6}}\right)\cos\mu\right)}{\sqrt{\frac{1}{6}}}$ … 1

Subs 1 into 2:

$\sqrt{\frac{1}{6}}(\cos\mu) - \sqrt{\frac{5}{6}}\left(\frac{\left(\frac{4}{5} - \left(\sqrt{\frac{5}{6}}\right)\cos\mu\right)}{\sqrt{\frac{1}{6}}}\right) = \frac{-3}{5}$

$\sqrt{\frac{1}{6}}(\cos\mu) - \frac{\sqrt{5}}{6}\left(\frac{4}{5} - \sqrt{\frac{5}{6}}\cos\mu\right) = \frac{-3}{5}$

$\sqrt{\frac{1}{6}}(\cos\mu) - \frac{2\sqrt{5}}{15} + \frac{5\sqrt{6}}{36}\cos\mu = \frac{-3}{5}$

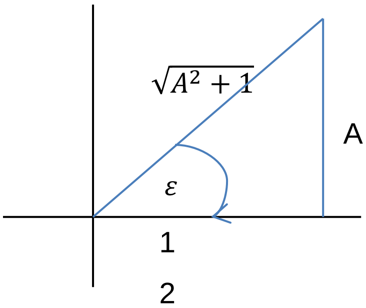
$\frac{11\sqrt{6}}{36}\cos\mu = \frac{-3}{5} + \frac{2\sqrt{5}}{15}$

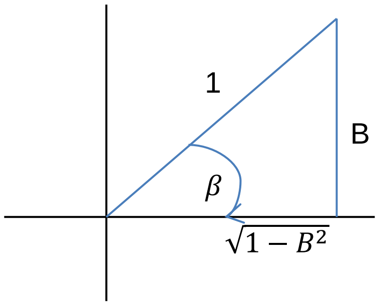
$\therefore \cos\mu = \frac{-9+2\sqrt{5}}{15} \div \frac{11\sqrt{6}}{36}$

$\therefore \cos\mu = \frac{4\sqrt{30}-18\sqrt{6}}{55}$

$\therefore \cos\mu = -0.403$

7. $A = \tan\varepsilon$ and $B = \sin\beta$





$r^2 = x^2 + y^2$

$r = \sqrt{1+A^2}$

$x^2 = r^2 - y^2$

$x = \sqrt{1-B^2}$

a) $\frac{\sin(\varepsilon+\beta)}{\sin 2\varepsilon}$

$= \frac{\sin\varepsilon\cos\beta + \sin\beta\cos\varepsilon}{2\sin\varepsilon\cos\varepsilon}$

$= \frac{\frac{A}{\sqrt{A^2+1}}\left(\frac{\sqrt{1-B^2}}{1}\right)+(B)\left(\frac{1}{\sqrt{A^2+1}}\right)}{2\left(\frac{A}{\sqrt{A^2+1}}\right)\left(\frac{1}{\sqrt{A^2+1}}\right)}$

$= \frac{\frac{A\sqrt{1-B^2}}{\sqrt{A^2+1}}+\frac{B}{\sqrt{A^2+1}}}{\frac{2A}{A^2+1}}$

$= \left(\frac{A\sqrt{1-B^2}+B}{\sqrt{A^2+1}}\right) \times \frac{A^2+1}{2A}$

$= \frac{\sqrt{A^2+1}\left(A\sqrt{1-B^2}+B\right)}{2A}$

b) $\sin\varepsilon \times \tan\beta$

$= \frac{A}{\sqrt{A^2+1}} \times \frac{B}{\sqrt{1-B^2}}$

$= \frac{AB}{\sqrt{(A^2+1)(1-B^2)}}$

$= \frac{AB}{\sqrt{A^2-A^2B^2-B^2+1}}$

c) $\cos 2\beta$

$= \cos^2\beta - \sin^2\beta$

$= \left(\frac{\sqrt{1-B^2}}{1}\right)^2 - \left(\frac{B}{1}\right)^2$

$= 1 - B^2 - B^2$

$= 1 - 2B^2$

d) $1 - \sin 2\beta$

$= 1 - 2\sin\beta\cos\beta$

$= 1 - 2\left(\frac{B}{1}\right)\left(\frac{\sqrt{1-B^2}}{1}\right)$

$= 1 - 2B\sqrt{1-B^2}$

8. a) $\sin\frac{C}{2}$

$\cos 2\left(\frac{C}{2}\right) = 1 - 2\sin^2\left(\frac{C}{2}\right)$

$\cos C = 1 - 2\sin^2\left(\frac{C}{2}\right)$

$\cos C - 1 = -2\sin^2\left(\frac{C}{2}\right)$

$-\frac{1}{2}(\cos C - 1) = \sin^2\left(\frac{C}{2}\right)$

$\therefore \sin\left(\frac{C}{2}\right) = \sqrt{-\frac{1}{2}(\cos C - 1)}$

b) $\cos\frac{D}{2}$

$\cos 2\left(\frac{D}{2}\right) = 2\cos^2\left(\frac{D}{2}\right) - 1$

$\cos D = 2\cos^2\left(\frac{D}{2}\right) - 1$

$\cos D + 1 = 2\cos^2\left(\frac{D}{2}\right)$

$\frac{1}{2}(\cos D + 1) = \cos^2\left(\frac{D}{2}\right)$

$\therefore \cos\left(\frac{D}{2}\right) = \sqrt{\frac{1}{2}(\cos D + 1)}$

c) $\tan 2E$

$= \frac{\sin 2E}{\cos 2E}$

$= \frac{2\sin E\cos E}{\cos^2 E - \sin^2 E}$

$= \frac{2\sin E\cos E}{\cos^2 E - \sin^2 E} \times \frac{\frac{1}{\cos^2 E}}{\frac{1}{\cos^2 E}}$

$= \frac{\frac{2\sin E\cos E}{\cos^2 E}}{\frac{\cos^2 E}{\cos^2 E} - \frac{\sin^2 E}{\cos^2 E}}$

$= \frac{\frac{2\sin E}{\cos E}}{1 - \frac{\sin^2 E}{\cos^2 E}}$

$= \frac{2\tan E}{1 - \tan^2 E}$

d) $\sin\frac{3}{2}F$

$= \sin\left(F + \frac{1}{2}F\right)$

$= \sin F\cos\frac{1}{2}F + \sin\frac{1}{2}F\cos F$

$= \sin F\left(\sqrt{\frac{1}{2}(\cos F + 1)}\right) + \left(\sqrt{-\frac{1}{2}(\cos F - 1\,)}\right)\cos F$